deformation shell theories. What is significant, however, is the difference in the stress distribution, and it is rather with this consideration in mind that future work in elasticity theory pertaining to reinforced shell design should be directed.

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Postbuckling Behavior of Axially Compressed Circular Cylinders

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The postbuckling behavior of a circular cylindrical shell subjected to axial compression is considered. Because of numerical difficulties, previous analyses of this problem have been severely restricted and do not yield quantitatively satisfactory results. The most accurate such analysis gives a minimum postbuckling load that is approximately three times higher than corresponding test values. In the present analysis, the number of free constants in the displacement function has been increased successively until no significant change occurs in the magnitude of the minimum postbuckling equilibrium load. The load so computed is found to be in close agreement with available test results. Although the minimum postbuckling load may be considered as a lower bound for the buckling load, it is, in general, too conservative to be practically useful as a design limit. Hence, the present analysis is not directly useful for design. However, it shows that agreement between theory and tests can be obtained for the postbuckling behavior of axially compressed cylinders, and it indicates which terms are needed in the assumed form of the deflection pattern in the finite displacement analysis. The analysis also can be extended to include, for instance, the effect of initial geometrical imperfections.

Nomenclature

a = cylinder radius
a_{ij} = constants [see Eq. (7)]
E = Young's modulus
F = stress function
j,k = integers [see Eq. (7)]
L = cylinder length
l_x,l_y = half wavelengths in axial and circumferential directions, respectively
n = number of waves in circumferential direction
t = shell thickness
u,v,w = nondimensional displacement components of a point in the middle surface of the shell in the axial, circumferential, and normal directions, respectively; corresponding distances are au, av, and aw, and w is positive inward
V = total potential energy of shell
x = nondimensional coordinate in axial direction; corresponding distance is ax
δ = end shortening per unit length
δ_{CL} = σ_{CL}/E
$\epsilon_x,\epsilon_\phi,\gamma_{x\phi}$ = strains at a point in the middle surface of the shell
$\sigma_x,\sigma_\phi,\tau_{x\phi}$ = stresses at a point in the middle surface of the shell, nondimensional
σ = axial compressive stress
σ_{CR} = critical compressive stress
σ_{CL} = $\{E/[3(1-\nu^2)]^{1/2}\}(t/a)$
ν = Poisson's ratio
ϕ = angular coordinate

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Introduction

It became evident early in the history of the problem that the classical theory fails to produce results in agreement with tests for the buckling of a cylindrical shell under axial compression. Hence, the theory of finite deflections has been explored as a possible way to obtain theoretical results in agreement with tests. The first such effort was made by Donnell,[1] and some more promising results were obtained by von Kármán and Tsien.[2] The theory gradually was refined in the following years.[3–5] Although these analyses fail to yield a useful design criterion, they contribute substantially to a better understanding of the problem. In particular, they indicate the presence of buckled equilibrium configurations corresponding to relatively low loads and thus the implication that the buckling load may depend to a great extent on such factors as accuracy in manufacturing, method of loading, and service environment.

Generally, the designer of axially loaded cylinders bases his decisions upon the results of the vast number of tests performed during recent decades. However, this is an unsatisfactory method, considering the multitude of influential parameters and the possibility that the available test results are not quite representative for the case to which they will be applied. For example, the case of dead-weight loading in general is applicable in practice, but very few of the existing tests approximate this condition. It is unfortunate that no better method is yet available, and the desirability of an advancement of the theory on the subject should be beyond question.

Finite Displacement Theory

The most significant contribution to the buckling theory for thin cylindrical shells is definitely the introduction of the finite displacement theory. The following discussion of this theory is applicable in part to the case of dead-weight loading only, but similar arguments can be made in an obvious way for any type of loading conditions.

It was shown by use of a finite displacement analysis that when the axial force exceeds a certain value—the minimum postbuckling load—a stable equilibrium form exists in addition to the membrane solution. It clearly is possible, whenever two stable equilibrium configurations exist, that the shell can be transformed from one to another, under the influence of some type of external disturbance. Hence, buckling or snap-through is conceivable when the shell is subjected to an axial force higher than the minimum postbuckling load. However, the two stable equilibrium configurations are separated by a barrier of higher energy, and it seems most unlikely that external disturbances are responsible for the wide scatter observed in tests that are performed in the relative quiet of a laboratory.

The presence of buckled equilibrium configurations corresponding to loads considerably below the classical buckling load also suggests possible sensitivity to initial imperfections. It was shown by Donnell and Wan[6] that this is indeed the case. It was assumed in their analysis that the most important factor affecting the buckling load of the shell would be the initial deviations from the ideal geometry, and, furthermore, that such deviations could be substituted for disturbances of any other kind. Although there are logical objections to the procedure, it also was assumed for simplicity that the geometrical deviations were proportional to the displacements under load. In this way, all disturbances could be lumped into one "unevenness parameter," U.

When the shape of the shell deviates from the true cylindrical form, no membrane solution exists. As the axial load is increased, the slope of the load deflection curve decreases, and a maximum load is reached (see Fig. 1). This maximum represents in the classical sense the buckling load of the shell, and, even for quite small initial imperfections, it is considerably below the classical buckling load for the geometrically perfect shell. A seemingly reasonable expression for U was used in the numerical analysis of Donnell and Wan,[6] leading to a buckling coefficient that depends on a/t in the same manner as is indicated by experiments.

However, the present finite displacement analyses are inaccurate. It is shown by Thielemann[7] that the minimum postbuckling load in the analysis of Kempner[5] is roughly three times higher than corresponding experimental values. Until this deficiency has been eliminated, the finite displacement theory cannot be a useful tool in the determination of buckling loads for axially compressed cylinders, whatever buckling criterion is finally chosen. It also is obvious that when the analysis is based on the initial imperfection approach, as in Ref. 6, little confidence can be placed in the results unless a sufficiently accurate displacement function is employed. The author's aims here are to show that the behavior of the shell under finite displacements can be predicted accurately and to find the corresponding displacement function.

Analysis

The postbuckling behavior of a thin-walled circular cylinder under axial compression is determined in the following analysis. It is assumed that the cylinder is of sufficient length so that the effects of end restraints are negligible. Only its elastic behavior is considered. The problem consists of finding equilibrium configurations in the postbuckling range and is approached in general with the same method as was used by previous investigators. The potential energy is expressed in terms of finite displacements, and equilibrium configurations are found by application of the principle of stationary potential energy.

Because of numerical difficulties, previous analyses were restricted to the use of displacement functions with very few degrees of freedom; with presently available high-speed computers, this obstacle is less formidable. The displacement functions therefore are expanded here, with the result that a load-deflection curve that is in agreement with test results is obtained. The analysis is based on the following basic equations:

Total Potential Energy

$$\frac{2}{E\cdot t\cdot a^2}\cdot V = \int_0^{L/a}\int_0^{2\pi}[(\sigma_x+\sigma_\phi)^2 - 2(1+\nu)\times (\sigma_x\sigma_\phi - \tau_{x\phi}^2)]dxd\phi + \frac{1}{12(1-\nu^2)}\cdot \left(\frac{t}{a}\right)^2\int_0^{L/a}\int_0^{2\pi}\left[\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial \phi^2}\right)^2 - 2(1-\nu)\times \left\{\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial \phi^2} - \left(\frac{\partial^2 w}{\partial x\partial\phi}\right)^2\right\}\right]dxd\phi - 2\int_0^{2\pi}(\sigma_x)_{x=L/a}\cdot d\phi\int_0^{L/a}\frac{\partial u}{\partial x}\cdot dx \quad (1)$$

Here σ_x, σ_ϕ, and $\tau_{x\phi}$ are nondimensional stresses at a point in the middle surface of the shell. They are defined by

$$\begin{aligned}\sigma_x &= [1/(1-\nu^2)](\epsilon_x+\nu\epsilon_\phi)\\ \sigma_\phi &= [1/(1-\nu^2)](\epsilon_\phi+\nu\epsilon_x)\\ \tau_{x\phi} &= [1/2(1+\nu)]\gamma_{x\phi}\end{aligned} \quad (2)$$

The quantities ϵ_x, ϵ_ϕ, and $\gamma_{x\phi}$ represent strains at a point in the middle surface of the shell. They presumably are expressed with sufficient accuracy in terms of displacement components in the following equations:

$$\begin{aligned}\epsilon_x &= (\partial u/\partial x) + \tfrac{1}{2}(\partial w/\partial x)^2\\ \epsilon_\phi &= (\partial v/\partial\phi) - w + \tfrac{1}{2}(\partial w/\partial\phi)^2\\ \gamma_{x\phi} &= (\partial u/\partial\phi) + (\partial v/\partial x) + (\partial w/\partial x)\cdot(\partial w/\partial\phi)\end{aligned} \quad (3)$$

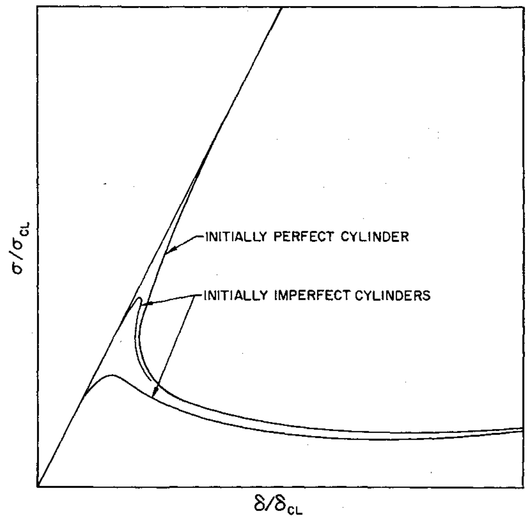

Fig. 1 Load-deflection curves for axially compressed cylinders

Two of the three equilibrium equations (Euler equations) posed by the variational problem (stationary V) are

$$(\partial\sigma_x/\partial x) + (\partial\tau_{x\phi}/\partial\phi) = 0$$
$$(\partial\tau_{x\phi}/\partial x) + (\partial\sigma_\phi/\partial\phi) = 0 \qquad (4)$$

These equations are satisfied identically if a stress function F is introduced so that

$$\sigma_x = \partial^2F/\partial\phi^2 \qquad \sigma_\phi = \partial^2F/\partial x^2$$
$$\tau_{x\phi} = -(\partial^2F/\partial x\partial\phi) \qquad (5)$$

Through combination of Eqs. (2, 3, and 5), a compatibility equation is obtained which can be written as

$$\nabla^4F = (\partial^2w/\partial x\partial\phi)^2 - (\partial^2w/\partial x^2)\cdot(\partial^2w/\partial\phi^2) - (\partial^2w/\partial x^2) \qquad (6)$$

where

$$\nabla^4 = [(\partial^2/\partial x^2) + (\partial^2/\partial\phi^2)]^2$$

Together with the requirement of stationary potential energy, Eqs. (1–3, 5, and 6) constitute a sufficient background for a solution of the problem under consideration.

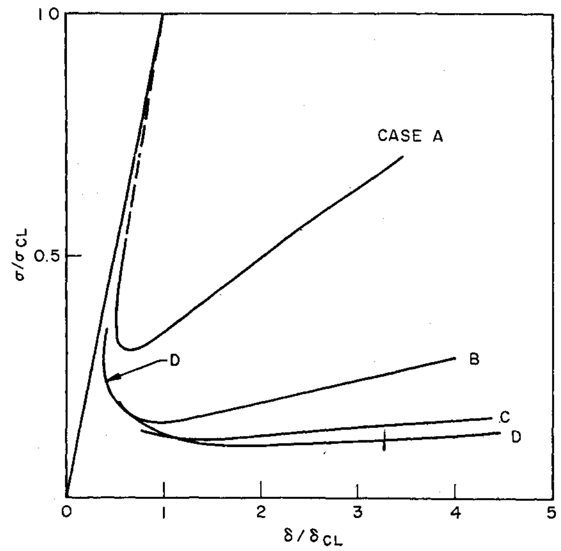

Fig. 2 Postbuckling curves

The successful solution of the problem requires the use of a sufficiently accurate displacement function. The normal displacement can be represented by a double Fourier series,

$$w = \sum_{j,k=0}^{\infty}\sum a_{jk}\cdot\cos\left(j\,\frac{a\pi x}{l_x}\right)\cdot\cos\left(k\,\frac{a\pi\phi}{l_y}\right) \qquad (7)$$

where l_x and l_y are axial and circumferential half-wave lengths. The diamond buckling pattern observed in tests indicates that only those terms need to be included in which $j + k = 0, 2, 4, 6. \ldots$ The most accurate of previous analyses[5] retains only a_{00}, a_{20}, a_{11}, and a_{02}. Terms will be added successively to the displacement function until no significant change occurs in the magnitude of the minimum postbuckling load. It should be noted that a_{00} is not a free constant but is determined by the condition that v must be a periodic function of ϕ.

By use of the equations given in the foregoing, the total potential energy V can be expressed in terms of l_x, l_y, and the a_{jk}'s. If the condition that $n = \pi\cdot a/l_y$ be an integer is waived, and if suitable substitutions are made,[5] it can be shown that the postbuckling function, $\sigma/E\cdot a/t$ vs $\delta\cdot a/t$, is

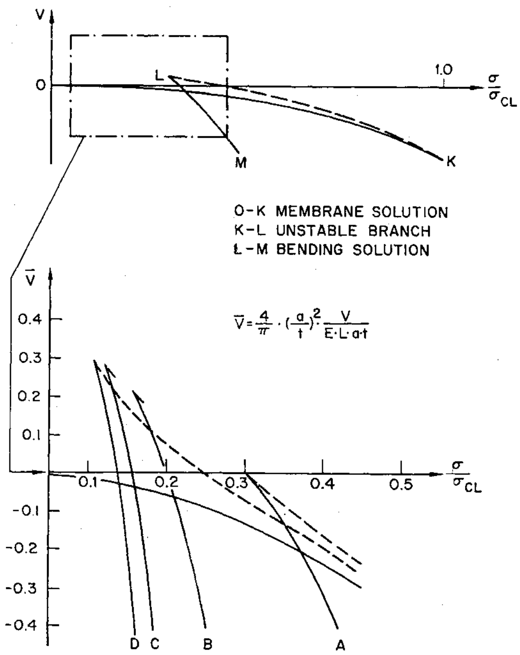

Fig. 3 Total potential energy vs stress

independent of a/t. The advantage of this procedure is obvious, and the approximation definitely is reasonable.

Minimization of the total potential energy with respect to the generalized coordinates of the shell (l_x, l_y, and the a_{jk}'s) yields a system of simultaneous nonlinear equations. These equations are given for case A by Kempner.[5] For the other cases, the equations are similar in nature, but with the increasing number of degrees of freedom they become quite long and will not be shown here. Solutions to these equations were obtained by use of the Newton-Raphson iterative method. This method is illustrated, for instance, by Hildebrand.[8] If the generalized coordinates are denoted by x_i, the equation system can be written

$$\partial V/\partial x_i = 0 \qquad i = 1, 2, \ldots N \qquad (8)$$

where N is the number of degrees of freedom of the system. Solutions to the equation system are found by use of the recurrence formula

$$\{x_i\}^{(k+1)} = (\{x_i\} - [\partial^2V/\partial x_i\partial x_j]^{-1}\{\partial V/\partial x_j\})^{(k)} \qquad (9)$$

where the superscripts (k) and $(k + 1)$ denote iteration number. In the neighborhood of the minimum postbuckling load, it is inconvenient to consider the stress as an independent variable. Difficulties in this region were avoided by use of an alternative formulation of Eq. (9), which is derived in an obvious way and allows the choice of the axial stress as a dependent variable in place of one of the generalized coordinates.

Numerical Results

In the numerical analysis, several different displacement functions were tried, but the complete postbuckling curve was computed for only a few. The free constants retained [see Eq. (7)] were as follows:

Case A: a_{20}, a_{11}, a_{02}

Case B: $a_{20}, a_{11}, a_{40}, a_{22}$

Case C: $a_{20}, a_{11}, a_{40}, a_{22}, a_{60}, a_{33}$

Case D: $a_{20}, a_{11}, a_{02}, a_{40}, a_{31}, a_{22}, a_{13}, a_{60}, a_{33}$

The load-displacement curves for these four cases are shown in Fig. 2. Case A reproduces the results of Kempner;[5] the other cases represent various degrees of refinement of that analysis. An attempt was made, also, to include all nine coefficients of case D but to omit some supposedly less influential terms in the potential energy expression. The results were not promising, and this type of procedure was not pursued further. The minimum postbuckling load corresponding to case D is given by $\sigma/E \cdot a/t = 0.0652$, which is 10.8% of the classical buckling load and is in close agreement with test results (10 to 12% of σ_{CL} according to Ref. 7). From the sizes of individual coefficients in case D and also from information obtained during the trials with other displacement functions, it was concluded that inclusion of additional terms probably would not lead to a significant change in the final results. It is seen that an accurate snap-through analysis should be based on a displacement function that at least contains the parameters used in case C (eight degrees of freedom, including the two wavelengths).

The number of waves at test is higher than is indicated by the present theory. However, it appears likely that the number of lobes formed at snapping is determined by conditions on the unstable branch. This explanation seems to be verified by tests on mylar cylinders, recently performed by Thielemann and Geier at Deutsche Forschungsanstalt für Luftfahrt, Braunschweig, Germany.[9]

Figure 3 shows the total potential energy of the shell vs axial stress. The size of the energy barrier for a fixed value of the stress can be read from this diagram as the difference between the energies of the two curve branches K-L and O-K. Information of this kind could be useful in design to resist, say, wind buffeting in service.

Concluding Remarks

The present analysis contributes a postbuckling load-displacement curve for axially loaded circular cylinders which is in agreement with experimental evidence. The minimum load in the postbuckling range may be considered as a lower bound for the buckling load of the cylinder, but it is, in general, too conservative to serve as a design limit. Although the analysis is not directly useful for design, it shows which terms are needed in the assumed form of the deflection pattern, and it can be extended, for instance, to include initial geometrical imperfections.

The ultimate goal of research in this field is, of course, the establishment of sound design principles for cylinders under axial compression. It is believed that this goal can be reached only after additional extensive analytical and experimental effort. The most critical need at present is believed to be for additional tests, which, unlike most previous tests, will be designed specifically to provide general information about the buckling process. Such tests could, for instance, show the effects of initial imperfections, loading procedure, and edge conditions.

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